

Environment-induced mixing processes in quantum walks

Lauri Lehman

JARA Institute for Quantum Information,
Otto-Blumenthal-Straße, RWTH Aachen, 52074 Aachen, Germany
llehman@physik.rwth-aachen.de

Abstract

The mixing process of discrete-time quantum walks on one-dimensional lattices is revisited in a setting where the walker is coupled to an environment, and the time evolution of the walker and the environment is unitary. The mixing process is found to be incomplete, in the sense that the walker does not approach the maximally mixed state indefinitely, but the distance to the maximally mixed state saturates to some finite value depending on the size of the environment. The quantum speedup of mixing time is investigated numerically as the size of the environment decreases from infinity to a finite value. The mixing process in this unitary setting can be explained by interpreting it as an equilibration process in a closed quantum system, where subsystems can exhibit equilibration even when the entropy of the total system remains zero.

1 Introduction

Quantum walks have emerged in recent years as a useful theoretical tool to study many fundamental aspects of quantum dynamics. The quantum walk model provides a platform to study the effects of decoherence and disorder at the microscopical level, in a simple setting that often allows the full analytical treatment of the problem. The running times of numerical calculations also scale favourably in the size of the system and in the running time of the walk, and large scale numerical calculations are therefore possible.

The notion of mixing processes originates from random walks, where the system's tendency to approach a completely random ("mixed") state over a long time period is called the mixing process. The completely mixed state at the end of the mixing process is characterized by a uniform probability distribution over the nodes of the graph, such that the walker can be found on any lattice site with equal probability. Since the final state is the same for any initial state, all the information about the initial position of the walker is lost, and the walk is said to have no memory (or to be Markovian). In unitary quantum walks, the total state of the system is always pure and the entropy is zero, so memory is preserved in the total wave function of the system. The system will explore all the possible configurations over time, and shows no tendency to approach a particular state such as the completely mixed state. If the walker is subject to decoherence, the unitarity is lost and the system evolves towards the completely mixed state.[1] Surprisingly, some amount of decoherence can even improve the mixing rate in quantum systems.[2, 4]

The process of decoherence in a quantum system is attributed to the existence of a coupling between the system and its environment. When the system and the environment interact with each other, they become entangled with each other, and a fully quantum mechanical treatment requires that both systems are described by density matrices, probabilistic mixtures of pure states. In

previous studies, the environment is usually described as a Markovian bath that interacts with the system with some probability, and the dynamics of the environment is completely neglected. Such models do not take into account the correlations between noise events which might arise when the environment has a memory. The Markovian nature of the environment in these models implies that any information about the initial state of the system is eventually lost, and the system approaches the maximally mixed state asymptotically. It is however interesting to ask how the memory of the environment affects the dynamics of the mixing process. In this article, the treatment of the environment is fully quantum mechanical, and the emergence of the mixing process is studied as the size of the environment increases.

The effects of decoherence in quantum walks have been studied extensively in the past[1]. The first studies[2] concentrated on the mixing properties of the time-averaged probability distribution, and studies on non-averaged walks followed.[3, 4] Recently, the concepts of quantum stochastic walks[5] and open quantum walks[6, 7] have been introduced, generalizing the formalism of quantum walks under decoherence, but mixing processes on finite graphs were not studied. Hines and Stamp[8] have studied effects of non-Markovian noise processes on continuous-time quantum walks, and found that in the presence of a non-Markovian bath, the propagation of the walker is ballistic. When the environment is described quantum mechanically as a spin bath, the mean-square displacement is reduced only by a constant factor, and even strong coupling with the environment does not induce diffusive behaviour in the long time limit, in stark contrast with Markovian noise models. Further similarities between the model introduced here and other models in the literature are discussed in the next section.

Here, the mixing properties of quantum walks are studied numerically using a discrete-time model with a finite number of sites. The walk takes place on a one-dimensional chain with periodic boundary conditions. When the Hilbert space of the environment is finite, the mixing behaviour of the walk was found to terminate after a certain number of time steps. During the initial mixing process, the distance to the maximally mixed state decreases exponentially, with a faster rate than the corresponding classical random walk. The mixing time of the walk was studied as a function of the size of the environment, and the mixing time was found to approach the classical value as the size of the environment increases. After the state has converged to a certain distance from the maximally mixed state, it saturates to an almost constant level, with only small fluctuations around the mean value. The average distance to the maximally mixed state in the long time limit has a power-law dependency on the ratio of the sizes of the environment and the lattice. These results hold when the coupling between the system and the environment is non-local, but studies on local environments show similar properties. The effect of non-commutativity of the environment matrices was also studied in a local model, and increasing the degree of non-commutativity was found to have similar effects as increasing the size of the environment.

2 Quantum walk with an environment

The transition of quantum states to classical states can be explained by decoherence of quantum states, and decoherence itself can be explained by interaction of the system with an environment.[9] To study how the existence of the environment induces mixing processes in quantum walks, let us introduce a generic model for a quantum mechanical environment. The states of the environment are written $|\epsilon\rangle = \sum_e \epsilon_e |e\rangle$ and belong to the Hilbert space $|\epsilon\rangle \in \mathcal{H}_E$, where the vectors $|e\rangle$ form an orthonormal basis for \mathcal{H}_E and the dimension of the space is denoted $\dim(\mathcal{H}_E) \equiv d_E$. The assumptions about the structure of the environment are kept to a minimum, in particular it is assumed that there are no symmetries associated with the environment.

The Hilbert space of the usual discrete-time quantum walk consists of position and coin. The full quantum state of the walker with an environment at time t can thus be written as $|\Psi(t)\rangle = \sum_{s,c,e} \psi_{sce}(t) |s\rangle |c\rangle |e\rangle$, where the vectors $|s\rangle \in \mathcal{H}_S$ and $|c\rangle \in \mathcal{H}_C$ form orthonormal bases for position and coin, respectively. The coin space is two-dimensional and the dimension of the position

space is denoted $\dim(\mathcal{H}_S) \equiv d_S$. The full Hilbert space is thus $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_C \otimes \mathcal{H}_E$, and the time evolution is unitary in the full space. The discrete time quantum walk evolves in discrete steps, with each step consisting of a coin flip and a conditional translation in position space.[10, 11] The coin flip is represented by a unitary matrix F , given by the Hadamard matrix:

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (1)$$

The conditional translation operators $T_- \otimes P_0$ and $T_+ \otimes P_1$, with $T_{\pm} = \sum_s |s \pm 1\rangle_S \langle s|$ and $P_c = |c\rangle_C \langle c|$, project the coin state to either $|0\rangle$ or $|1\rangle$, accompanied by a translation to the neighbouring site on the left or right, respectively. To include the coupling between the system and the environment, some additional dynamics must be introduced such that the system becomes entangled with the environment. For this purpose, introduce two unitary matrices E_0 and E_1 that transform the state of the environment at each time step, in a conditional way such that E_0 is applied for left moving components and E_1 for right moving components. The time evolution for one time step of the walk is now given by the unitary time evolution operator U :

$$|\Psi(t+1)\rangle = U|\Psi(t)\rangle \quad (2)$$

$$U = T_- \otimes P_0 F \otimes E_0 + T_+ \otimes P_1 F \otimes E_1 \quad (3)$$

with periodic boundary conditions assumed at the ends of the chain. The number of sites d_S is always chosen to be odd, so that the walker can be found on any site at any given time step, and not just on even or odd sites. In the beginning of the walk, the wave function is initialized in the product state $|\Psi(0)\rangle = |s_0\rangle|c_0\rangle|\epsilon_0\rangle$, with $|s_0\rangle$ localized with unit probability on one of the sites on the ring. The initial state of the coin and the environment are not relevant for the mixing process, as discussed below. The final state at time step t is obtained simply by multiplying the wave function by U iteratively: $|\Psi(t)\rangle = U^t|\Psi(0)\rangle$. The main object of interest is the reduced density matrix of position space, which is obtained by tracing over the coin and environment DOFs: $\rho_S(t) = \text{Tr}_{CE}(\rho(t))$, where the total state is always pure, $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$.

The matrices E_0 and E_1 are assumed to be random unitary matrices acting in the Hilbert space of the environment \mathcal{H}_E . The most important property of these matrices is that they should not be commuting, $[E_0, E_1] \neq 0$. Only in this case does the environment have a non-trivial effect on the dynamics. It can be shown that if the matrices do commute, the coupling with the environment has no effect on the spatial probability distribution, and only the non-diagonal values of the spatial density matrix ρ_S acquire a position-dependent complex phase. The random unitary matrices E_c can be constructed in a variety of ways. Here we take an elementary approach by picking a random Hermitian matrix H_c with elements inside a fixed region around $0 + 0i$, and exponentiating the matrix: $E_c = \exp(-iH_c)$. The matrices H_c can thus be viewed as the generating Hamiltonians for the dynamics of the environment, with time unit $\Delta t = 1$.

Since the total wave function now includes the environment DOFs, the evolution of the walker in position and coin space is no more unitary. To obtain the state of the walker in position and coin space, the environment DOFs must be traced out, and if the walker is entangled with the environment, the resulting state is a mixed state. The effect of the environment is thus that it induces a completely positive (CP) map on the quantum walker, with the number of decoherence channels equal to the dimension of the environment. Denoting the reduced density matrix of position and coin by $\rho_{SC}(t)$, the effect of the CP map \mathcal{E} can be written formally as

$$\rho_{SC}(t) = \mathcal{E}(\rho_{SC}(0)) = \sum_{e=1}^{d_E} X_e \rho_{SC}(0) X_e^\dagger \quad (4)$$

where the matrices X_e are the Kraus generators of the CP map acting in the space $\mathcal{H}_S \otimes \mathcal{H}_C$ and satisfying $\sum_e X_e^\dagger X_e = \mathbb{I}_{SC}$. Their expressions can be given as

$$X_e = \left(\mathbb{I}_{SC} \otimes \langle e| \right) U^t \left(\mathbb{I}_{SC} \otimes |\epsilon_0\rangle \right) \quad (5)$$

where the vectors $|e\rangle$ form an orthonormal basis for \mathcal{H}_E , \mathbb{I}_{SC} is the identity transformation in space $\mathcal{H}_S \otimes \mathcal{H}_C$, and $|\epsilon_0\rangle$ is the initial state of the environment. Note that the CP map \mathcal{E} describes the time evolution from the initial state to the final state at time t . This is in contrast to previous studies of mixing processes,[3, 12] where the CP map describes the time evolution from time step t to $t + 1$. The previous studies assume implicitly that the noise acts in the same way at each time step, neglecting the time correlations of noise events which arise when the environment has a memory.

The quantum walk model with an environment is similar in spirit to quantum walks with multiple coins, first studied by Brun *et al.*[13] In these models the coin is still two-dimensional and the total evolution is unitary, but the coins are picked from a finite set at each time step and are reused cyclically, and the changing of the coins can be thought to be done by the environment. The variance of quantum walks with multiple coins behaves initially like the classical random walk, but changes to quadratic behaviour of the standard quantum walk after the initial classical period. Numerical calculations show that the model introduced in this section behaves in a similar way when considered on an infinite line. Increasing the number of coins in the model of Brun *et al* increases the length of the classical period in the beginning; similarly increasing the size of the environment d_E increases the duration of classical propagation in the beginning.

The quantum walk with an environment also bears resemblance to quantum walks with non-Abelian anyons,[14, 15] where one mobile anyon moves in a chain of stationary anyons, braiding with the stationary anyons as it hops between the lattice sites. There the anyons possess a collective fusion Hilbert space, and braids between the anyons induce non-trivial transformations in the fusion space. While the fusion space can also be viewed as a kind of environment, the quantum walks with anyons were only studied on infinite chains, and the mixing processes in finite chains were not considered.

In the following, the total system is divided into the system S and its bath B , where the system consists of the position DOFs and the bath consists of the coin and environment DOFs together. The total Hilbert space is thus decomposed as $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ with $\mathcal{H}_B = \mathcal{H}_C \otimes \mathcal{H}_E$, and the dimension of the bath is $d_B = 2d_E$. This notation makes explicit the fact that the coin and the environment together can be viewed as a big coin with enlarged dimension.

Although this model can be viewed as a quantum walk with redundant coin DOF's, the mixing behaviour is specific to this model. A quantum walk with a multidimensional coin, such as a discrete Fourier transform or Grover coin, with coin projection operators P_0 and P_1 defined as projections to two orthogonal subspaces, does not exhibit mixing, except in the average sense, like the usual quantum walk with a 2-dimensional coin.

3 Mixing time

In the classical random walk, the state of the system evolves always towards the completely mixed state, if there is no memory in the system. In the unitary quantum walk, the coin can be thought to act as a memory of the system, and the probability distribution does not approach any particular state (although the time-averaged probability distribution can be shown to approach a certain state[11]). If the quantum walk is subject to decoherence acting via random noise events, the probability distribution is known to approach the completely mixed state.[3, 4] Here we assess a different scenario where the decoherence is caused by the presence of an environment. Now the total wave function of the system evolves unitarily, the information about the state of the environment is kept for the whole duration of the walk, and the noise events at different time steps are correlated with each other. As discussed below, the mixing process is qualitatively different in this case. When the size of the environment is finite, the mixing process terminates after a certain time. Once the walker has converged within a certain distance from the completely mixed state, the distance does not decrease anymore, and the walker stays at almost constant radius from the completely mixed state.

For quantum systems, the completely mixed state is a diagonal density matrix, denoted ω , such that all the diagonal values are equal to each other: $(\omega)_{s,s'} = \delta_{s,s'}/d_S$. The degree of mixing (in position space) can be quantified by the distance to the maximally mixed state $D_\omega(t)$, given by

$$D_\omega(t) = D(\rho_S(t), \omega) \quad (6)$$

where the trace distance between two density matrices is defined as

$$D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} \sqrt{(\rho_1 - \rho_2)^2}. \quad (7)$$

During the mixing process, the distance $D_\omega(t)$ decreases as the state approaches the mixed state ω . In a random walk or in a quantum walk subject to random noise events, the walker approaches the state ω asymptotically and $D_\omega(t) \rightarrow 0$ as $t \rightarrow \infty$.

The time evolution of the reduced density matrix $\rho_S(t)$ was studied numerically for various sizes of the chain and the bath, and also for various initial states, coin matrices F and environment matrices E_c . Generally, the time evolution can be divided into two regions, the initial relaxation regime and the stabilized regime. In the initial stage of the walk, the system is mixing (or “equilibrating”) and D_ω decreases exponentially in time. When D_ω has reached a certain value, the mixing process ends, and D_ω stays almost constant in time. The remainder of this section concentrates on the analysis of the initial mixing process, while the stabilized regime is treated in the next section.

Typical results for the time evolution of the distance $D_\omega(t)$ are shown in Fig. 1 a), along with the results for the classical unbiased random walk. These results are obtained for a specific choice of parameters, without any averaging over initial states, environment matrices, or time. Already at modest size of the environment $d_B = 256$, the evolution is close to the classical behaviour. As the size of the environment decreases, the slope becomes slightly steeper, indicating that the mixing process becomes faster. This can be viewed as a quantum speedup of the mixing process. Also visible at $d_B = 64$ is the offset of the mixing process: as the distance reaches the value of $D_\omega(t) \approx 0.28$, it saturates to an almost constant level. This also happens when the size of the bath is larger, but the threshold of distance where this happens is smaller when d_B gets larger.

Similar behaviour is also observed for the von Neumann entropy, $H(\rho_S) = -\text{Tr}(\rho_S \ln \rho_S)$ (not plotted here). In the initial stage of the walk, the entropy H increases, but after reaching a certain value it saturates to an almost constant level. When the size of the environment increases, the saturated value of H approaches the maximal value $H_{\max} = \ln d_S$, which is the asymptotic value for a classical random walk.

Since the quantum walk with an environment does not approach the mixed state ω indefinitely, the usual notion of mixing time is problematic, because the state might never converge within a predetermined distance from ω . A more suitable measure of mixing time is given by the slope of the exponential curve:

$$D_\omega(t) \propto e^{-\frac{t}{\tau_{\text{mix}}}}. \quad (8)$$

Defined in this way, the mixing time determines the time scale at which the distance D_ω decreases by a factor of $1/e$. The mixing time can then be calculated from an exponential fit for the relevant time region $t_1 < t < t_2$. Two examples of fitted curves are drawn as black lines in Fig. 1 a).

To understand how the mixing time changes as a function of space and bath dimensions, the time evolution of the system was studied numerically for various configurations. The results for the mixing time τ_{mix} are shown in Fig. 1 b). Each point represents a quenched average over the environment: the walk was run with different environment matrices, but keeping the same matrices during each run (with a fixed initial state and coin). When the size of the bath grows, the dynamics starts to resemble the classical random walk, and the mixing time converges to the classical value given by the horizontal lines for each system size. In the intermediate region $d_B/d_S \approx 5$, the mixing time scales roughly logarithmically in the size of the bath, although the numerical data are

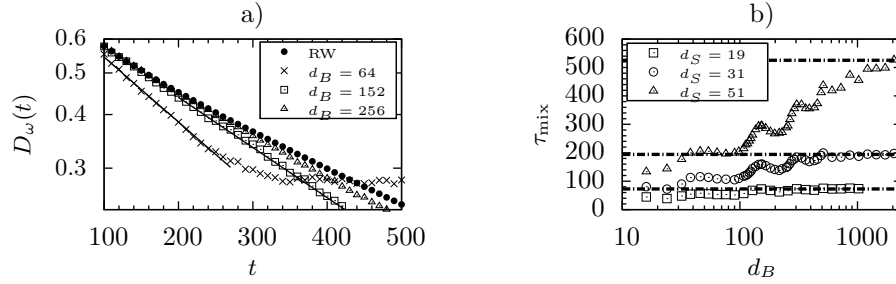


Figure 1: a) Typical time evolution of the distance $D_\omega(t)$ during the initial relaxation period for different choices of the bath dimension d_B (no averaging). ‘RW’ stands for classical random walk with equal hopping probabilities. The total number of sites is $d_S = 51$. Note that the scale on the y-axis is logarithmic. The black lines indicate the best fit to an exponential function. b) Quench-averaged mixing time as a function of the bath dimension d_B . Each point is averaged over 30 samples, and the error bars (not shown) are almost completely covered by the point marker. The horizontal lines show the mixing time of the classical random walk for each lattice size.

not conclusive enough to decide between logarithmic and power-law dependence. It can also be seen that the scaling function of the mixing time includes an additional periodic component, causing the oscillations which can be seen particularly at large system sizes.

The behaviour of the mixing time in Fig. 1 b) can be viewed as a transition between quantum and classical properties of the walk, with the classical limit approached as $d_B \rightarrow \infty$. The transition is rather smooth, so that increasing the size of the bath by a small amount means that the dynamics is also a little bit closer to the classical dynamics, although the oscillatory behaviour causes some deviations. Approximately logarithmic scaling of the mixing time implies that there is a large region of intermediate bath sizes where there is some amount of quantum speedup. On the other hand, logarithmic scaling would mean that decreasing the size of the bath does not reduce the mixing time efficiently. Indeed, as seen in Fig. 1 a), decreasing the size of the environment from infinity down to $d_B = 256$ changes the slope by a very modest amount. Generally the speedup is more dramatic when the size of the lattice is large. Comparing the speedup when the size of the environment decreases from the classical value ($d_B \rightarrow \infty$) to $d_B \approx 10$ in Fig. 1 b), the mixing time is approximately halved when $d_S = 31$, while for $d_S = 51$ it reduces to one quarter of the classical value.

The time evolution of the walk is almost insensitive to the initial state of the bath, and the changes for different initial states are of the same order as for different environment matrices (with fixed dimension). In contrast, the changes for different initial states of position can be drastic. Starting from a non-localized state has essentially the same effect as increasing the size of the bath, and the distance D_ω saturates to a higher level at long time scales when the walker starts from a superposition of several sites (only pure initial states were considered). Also the choice of the coin can affect the dynamics. While the long-time state is very similar for different choices of the coin, the speed of the initial mixing process can vary for different coins, the Hadamard coin being generally slower than other choices.

4 Stabilized dynamics

As seen in Fig. 1 a) for $d_B = 64$, the distance D_ω stops to decrease after it has reached a certain value, and remains at almost constant level from then on, with only small fluctuations around the mean value. When the size of the bath increases, the average distance becomes smaller and the

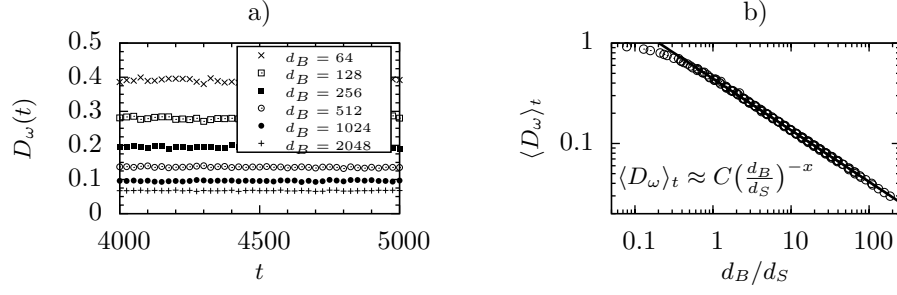


Figure 2: a) Distance to state ω after the termination of the initial mixing process. Depending on the bath size, distance D_ω saturates to a certain average level. The lattice size is $d_S = 51$. b) Average distance in long-time dynamics as a function of d_B/d_S . The lattice sizes are $d_S = \{11, 19, 31, 51\}$. See the main text for numerical estimates of the parameters C and x .

walker stays closer to the maximally mixed state ω . It is thus natural to ask how the average distance changes as a function of the bath size d_B .

The stabilized regime begins when the distance has saturated close to the long-time average. As seen in Fig. 2 a), fluctuations around the mean value are small when t is large, and the average distance shows a clear dependency on the size of the bath. To understand how the average distance depends on the bath size, define the long time average of D_ω as

$$\langle D_\omega \rangle_t = \frac{\sum_{t'=t_0}^t D_\omega(t')}{t - t_0 + 1} \quad (9)$$

where t_0 is large enough so that D_ω is close to the long-time average. The time average $\langle D_\omega \rangle_t$ is plotted in Fig. 2 b) for different choices of d_S and d_B . When plotted against the ratio d_B/d_S , the points for different lattice sizes fall on the same line, showing that the long-time average depends only on the ratio d_B/d_S .

The shape of the function in Fig. 2 b) suggests that the average distance has a power law dependency on the ratio d_B/d_S :

$$\langle D_\omega \rangle_t \approx C \left(\frac{d_B}{d_S} \right)^{-x} \quad (10)$$

when $d_B > d_S$. The numerical coefficients obtained by fitting are $C = 0.4411 \pm 0.0013$ and $x = 0.5133 \pm 0.0013$, and the corresponding function is drawn as a black line in Fig. 2 b).

The average distance is not the only property of stabilized dynamics which is sensitive to the size of the bath. Also the fluctuations around the average value change with d_B , and the variance of the fluctuations becomes smaller as d_B increases. The effect of the environment is therefore that it drives the state of the system to a small subset of all possible states, with the distance to the maximally mixed state restricted to a certain interval. When the size of the environment increases, the interval of allowed values becomes smaller and the system is driven to a more restricted subset of all states.

Eventually, the recurrence effect should bring the state close to its initial state, and the distance to the maximally mixed state should increase. The recurrence time in this system appears to be quite long however, and the system stayed close to its long-time average for all cases studied. In a few very long calculations, no recurrence was observed during 100 000 time steps of time evolution.

5 Local environments

In the previous description of the environment, it was assumed that the environment acts in the same way for every position of the chain. Thus the description of the environment is non-local, as the walker sees the same environment at every position. In a more realistic model, each site should have their own local environment, such that the walker interacts with the environment only when it is located in the respective position. The mixing process was also studied by considering such local environments.

A simple model for a local environment can be constructed by attaching an ancillary qubit to each lattice site. When the walker occupies a given site, it interacts only with the qubit attached to that site. The environment consists of d_S qubits, with dimension $d_E = 2^{d_S}$, and the environment matrices carry also a position label. The time evolution operator U is now given by the expression

$$U = \sum_s \left(T_-^s \otimes P_0 F \otimes E_0^s + T_+^s \otimes P_1 F \otimes E_1^s \right) \quad (11)$$

where the site-dependent translation operator is written as $T_\pm^s = |s \pm 1\rangle_S \langle s|$, and the matrix E_c^s acts on qubit s by a 2×2 matrix and trivially on other qubits. Each qubit rotation can be parametrized using two real variables θ and γ . The environment matrices can be written as $E_c^s = \mathbb{I}^{\otimes(s-1)} \otimes G_c \otimes \mathbb{I}^{\otimes(d_S-s)}$, where the unitary matrix G_c is given by

$$G_c = \begin{pmatrix} \cos \theta_c & -e^{-i\phi_c} \sin \theta_c \\ e^{i\phi_c} \sin \theta_c & \cos \theta_c \end{pmatrix}. \quad (12)$$

The environment is assumed to be homogeneous and the same matrices G_c are applied at every site.

It was mentioned above that if the matrices E_0 and E_1 commute, the environment has no effect on the probability distribution. The extent to which these matrices fail to commute can thus be seen as a measure of how strongly the system and the environment are coupled, and intuitively the growth rate of entanglement is higher when the matrices are less commuting. To investigate how the mixing process is affected by the non-commutativity, define the parameter γ as the matrix norm of the commutator:

$$\gamma = \| [G_0, G_1] \|. \quad (13)$$

Parametrization of the matrices according to Eq. (12) ensures that the parameter γ can be varied in a systematic way.

The time evolution of D_ω in the local model is plotted in Fig. 3. The effect of increasing the parameter γ in Fig. 3 b) is similar to increasing the size of the environment in Fig. 3 a): the system saturates to a state which is closer to the mixed state ω . The time evolution during the initial mixing process is too jumpy to determine how the mixing time changes as a function of γ . The large distance fluctuations in Fig. 3 are probably just a finite size effect of the lattice size, because similar fluctuations are also seen in the non-local model for the corresponding sizes of the lattice and the environment. Unfortunately the dimension of the environment grows exponentially in the number of sites, and numerical calculations become difficult for larger systems.

Intuitively one might expect that the parameter γ would act only to slow down the mixing process, but in the long time limit D_ω would saturate to the same value for all values of the parameter γ . The differences were however found to persist for very long times. Even after 50 000 time steps, the mean distances have not changed significantly from Fig. 3 b).

The local model exhibits essentially the same properties as the non-local model. When the size of the environment is larger, the system saturates closer to the maximally mixed state. Changing the initial state of the bath affects only the details of time evolution, not the asymptotic state, but starting from an initial state which is spread out in position space causes the walker to remain in

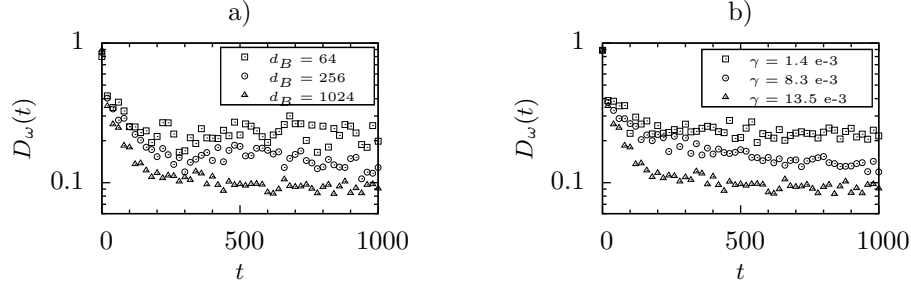


Figure 3: a) Time evolution of D_ω in a local environment for different bath sizes. The bath sizes are $d_B = \{64, 256, 1024\}$ corresponding to lattice sizes $d_S = \{5, 7, 9\}$, respectively. The environment matrices E_c were chosen such that $\gamma = 13.5 \text{ e-3}$. b) Time evolution of D_ω in a local environment for different values of γ . The parameter γ quantifies the noncommutativity of the environment matrices, see Eq. (13). The lattice size is $d_S = 9$ and the bath dimension is $d_B = 1024$.

state with a higher distance to ω . Comparing the local and non-local models with equally sized systems and baths, the non-local model is more “classical”, in the sense that the non-local walk saturates closer to ω than the local walk.

6 Conclusions

The emergence of the mixing process in quantum walks was studied in a finite-dimensional setting, where the walker becomes entangled with an environment, and the system-environment dynamics is unitary. The time evolution was found to be divided in two distinct regimes. In the initial regime of time evolution, the distance to the maximally mixed state decreases exponentially, and the rate of mixing is higher when the size of the environment is smaller. This enhancement in mixing time can be viewed as a quantum speedup, and the mixing time converges to the classical value as the size of the environment approaches infinity. When the size of the environment is finite, the duration of the mixing process is also finite. After the state has converged to a certain distance from the maximally mixed state, it ceases to approach the mixed state and saturates to an almost constant distance. The mean distance (and fluctuations around the mean) in this stabilized regime of time evolution also depend on the size of the environment, the distance becoming smaller as the size of the environment increases. The mean distance was found to follow a power law as a function of d_B/d_S , the ratio between the sizes of the bath and the system.

These results are in contrast with earlier results on mixing processes in quantum walks,[2, 3, 4] where the walker was taken to be subject to random uncorrelated noise events. Those models assumed implicitly that the environment is Markovian, while the unitary dynamics employed in this study preserves the memory of the environment. It has been shown here that when the size of the environment is finite, the memory can induce qualitatively different behaviour in the long time limit, and the state of the system tends to fluctuate around an equilibrium state which is not the maximally mixed state.

The dynamics of the mixing process can be understood in light of some recent results in quantum information theory and quantum thermodynamics. Linden *et al*[16] have shown that in a large unitarily evolving quantum system, any small subsystem tends to equilibrate eventually to a particular state, given that the initial state has support over sufficiently many eigenstates of the Hamiltonian. In the context of quantum walks, this translates to the requirement that the initial state $|\Psi(0)\rangle$ should have support over many eigenstates of the time evolution operator U . This condition was fulfilled by all the cases considered in this article.

The mechanism of equilibration in closed quantum systems is related to the surprising fact that in a randomly chosen pure quantum state, most subsystem states are almost maximally entangled for some bipartitioning of the system. Hayden *et al*[17] have proved some rigorous bounds for the distance between a typical state and the maximally entangled state, when both subsystems are large. The key idea is that in unitary evolution the system samples the states in a uniform way over time, and since most states are highly entangled, the small subsystem spends most of the time close to the maximally mixed state.

To understand the systems' tendency to be closer to the maximally mixed state when d_B is large, it is instructive to look at the average entropy of a quantum subsystem. Partitioning a randomly chosen pure state in two systems with dimensions $d_S \leq d_B$, the average entropies of the subsystems are [18, 19] $\langle H \rangle = \sum_{k=d_B+1}^{d_S d_B} 1/k - (d_S - 1)/2d_B$. Keeping the smaller dimension d_S fixed, the average entropy increases when d_B increases, and the entropy approaches the maximum value $\langle H \rangle \rightarrow \ln d_S$ as $d_B \rightarrow \infty$. A higher entropy means that the system is closer to the maximally mixed state, and therefore the distance D_ω is smaller when d_B is large.

This work might pave way for engineering mixing processes with a high level of control on their properties. The mixing time is a standard benchmark quantity for many random walk based algorithms, and understanding the conditions for improving the mixing time is essential. The mixing process is also essential for quantum random number generators, which require that random numbers are drawn from a distribution that is as uniform as possible. By carefully tuning the interaction between the sampled system and its environment, it could become possible to engineer the distribution of random variables such that they can be guaranteed to be close to the completely mixed distribution.

It is generally an interesting question to determine the conditions when the mixing process is induced by the environment. As discussed above, simply enlarging the coin dimension does not always lead to mixing behaviour. It would also be desirable to have analytical expressions for the equilibrium state and its distance from the maximally mixed state ω , and for the standard deviation of the fluctuations around the equilibrium state. Finally, it would be interesting to consider how often the system exhibits recurrences, where the distance D_ω jumps close to its initial value. As the size of the system and the environment increases, most states are close to the maximally mixed state [17], and product states such as the initial state $|\Psi(0)\rangle$ are more unlikely, so it is expected that recurrences are rare in large systems.

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